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"Study of Modern Instrumentation and Methods for Astronomic Positioning in the Field"

FIRST TECHNICAL REPORT

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0. Introduction

The traditional task of geodetic astronomy as seen from the viewpoint of practical geodesy is the determination of three spatial orientation parameters of a vertically set up observation instrument relative to a global reference frame fixed in the earth. For this horizontal and vertical directions are measured at registered instants to stars, whose coordinates are assumed to be known in a space fixed reference frame, the time dependent orientation of which, relative to the earth fixed frame, is also presumed to be known.

The required parameters are the *astronomical longitude* Λ and the *astronomical latitude* Φ, which fix the direction of the local gravity vector relative to the earth fixed reference frame, and the horizontal orientation unknown Σ of the instrument, which yields the astronomical azimuths A in connection with the measured horizontal directions to terrestrial objects.

A spatial reference frame is here understood as a triad of orthonormal base vectors, which is fixed to a distinct origin point and which is taken as being rotationally and translationally invariable in time.

The models used traditionally in geodetic astronomy are described in the standard textbooks (for example, K. Ramsayer, 1970, see Appendix A.1) mainly by making use of spherical trigonometry. The presentation used in this report is based upon orthonormal triads of base vectors of different reference frames. A compact and strictly systematical presentation is obtained by a *commutative diagram* of transformations between the respective bases, which is introduced in chapter 1. Fundamental relations between the parameters of geodetic astronomy result from this commutative diagram. In chapter 2 these relations are transformed after the linearization with suitable approximate values in such a way that a system of condition equations with unknowns ensures, the *Gauß-Helmert model* of the adjustment calculation. This system is specified for different combinations of observations. In chapter 3 the accuracies in the determination of Λ,Φ and Σ, which are expected for different configurations of stars, are estimated in simulation studies according to the Gauß-Helmert model and then the results are illustrated in diagrams.

The underlying principles for this report were taken from the dissertation of B. *Richter*, "Entwurf eines nichtrelativistischen geodätisch-astronomischen Bezugssystems", Deutsche Geodätische Kommission Heft C322, Munich, 1986, and from the manuscript of the lecture "Geodetic Astronomy", which B. Richter gives at the University of Stuttgart, Federal Republic of Germany.

1. Relations between the reference frames in geodetic astronomy

First of all in this chapter the fundamental relations in geodetic astronomy between observations, unknowns and given coordinates shall be derived which will be needed further.

1.1 The systematical structure of the reference frames

The reference frames used in geodetic astronomy may be arranged on different levels which are numbered in turn or indicated by symbols: 0 corresponds to ', 1 to *, 2 to •, 3 to ∘. One fundamental vector $\underset{\sim}{V}^i$ belongs to every level i. In details this is as follows:

$\underset{\sim}{V}^0 = \underset{\sim}{V}' = \underset{\sim}{Z}$	the position vector from the point of observation to the target object (terrestrial or celestial), which is generally a star;
$\underset{\sim}{V}^1 = \underset{\sim}{V}^* = -\underset{\sim}{\Gamma}$	the negative gravity vector;
$\underset{\sim}{V}^2 = \underset{\sim}{V}^\bullet = \underset{\sim}{\Omega}$	the earth rotation vector (it has the direction of the axis of the earth, points to the North Pole and has the value of the earth rotation rate);
$\underset{\sim}{V}^3 = \underset{\sim}{V}^\circ = \underset{\sim}{\psi}$	the ecliptic normal vector (it points to the northern pole of the ecliptic).

An orthonormal reference frame $\underset{\approx}{E}^i$ belongs to every level with its base vectors as follows:

$$\underset{\sim}{E}_3^i = \text{norm}\ \underset{\sim}{V}^i \tag{1-1}$$

$$\underset{\sim}{E}_2^i = \text{norm}\ (\underset{\sim}{V}^{i+1} \times \underset{\sim}{V}^i) \tag{1-2}$$

$$\underset{\sim}{E}_1^i = \underset{\sim}{E}_2^i \times \underset{\sim}{E}_3^i \tag{1-3}$$

Here "norm" denotes the abbreviation for the normalization of a vector, and "x" the vector product.

New reference frames are at the lower end the observational frame $\underset{\approx}{E}'$ of the level

"0", whose third base vector is located in the direction of the observation and which is unique since it is not a reference frame in the literal sense, because there are no vectors described with regard to this frame, and at the upper end the ecliptic frame $\underset{\approx}{E}^{o}$, which has hardly any practical importance.

In addition to the systematical $\underset{\approx}{E}$-triads, $\underset{\approx}{F}$-triads also appear on each level. These systems have the common third base vector with the appertaining $\underset{\approx}{E}$-frame, nevertheless the direction of the first and second base vector does not follow from the systematic structure of the fundamental vectors $\underset{\sim}{V}^{i}$, $\underset{\sim}{V}^{i+1}$, but from a more or less arbitrary definition.

A new $\underset{\approx}{F}$-triad is the theodolite frame $\underset{\approx}{F}^{*}$, whose first base vector $\underset{\sim}{F}_{1^*}$ lies in the direction "zero" of the azimuth circle of a theodolite which is set up in the astronomical horizon. The longitudinal angle (see below) of an observed direction in the local horizon frame is the horizontal direction T and is recorded systematically *in* the clockwise direction, but conventionally *counter*-clockwise, $T_s = -T_c$. The latitude angle (see below) is the vertical direction as in the horizon frame $\underset{\approx}{E}^{*}$.

The transformation from a frame $\underset{\approx}{E}^{i}$ to the appertaining frame $\underset{\approx}{F}^{i}$ is always a counter-clockwise rotation round the common third axis with the orientation angle H^{i},

$$\underset{\approx}{F}^{i} = \underline{R}_3(H^{i})\underset{\approx}{E}^{i} \ . \tag{1-4}$$

$\underline{R}_3$ is the rotation matrix, which describes a rotation of a frame round the third axis. It is

$$\underline{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1-5}$$

Corresponding to eqn. (1-5) the rotation matrices for the rotations round the first and second axis read

$$\underset{\sim}{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{1-6}$$

$$\underset{\sim}{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \tag{1-7}$$

$\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$ and $\underset{\sim}{R}_3$ are also called elementary rotations. The orientation angles H^i (see eqn. (1-4)) are in detail:

$H^1 = H^* = \Sigma$	the orientation unknown of the theodolite which has been set up;
$H^2 = H^{\bullet} = \theta_{Gr,s}$	the Greenwich sidereal time;
$H^3 = H^{\circ}$	the angle between the line of intersection of the ecliptic with the mean galaxy plane and the direction to the vernal equinox ±90°.

For the transformation from a frame $\underset{\approx}{E}^{i+1}$ to the underlying frame $\underset{\approx}{E}^{i}$ one needs the longitudinal angle χ^i_{i+1} and the latitude angle Φ^i_{i+1} of the fundamental vector $\underset{\sim}{V}^i$ with regard to the frame $\underset{\approx}{E}^{i+1}$:

$$\begin{aligned} \underset{\approx}{E}^i &= \underset{\sim}{R}_2(90°-\Phi^i_{i+1}) \; \underset{\sim}{R}_3(\chi^i_{i+1}) \; \underset{\approx}{E}^{i+1} = \underset{\sim}{R}_E(\chi^i_{i+1},\Phi^i_{i+1},0)\underset{\approx}{E}^{i+1} \\ &= \underset{\sim}{R}_E(\chi^i_{i+1},\Phi^i_{i+1})\underset{\approx}{E}^{i+1} \quad . \end{aligned} \tag{1-8}$$

$\underset{\sim}{R}_E$ is the special case of a rotation matrix of Eulerian type, in which the three elementary rotations are connected in a row as follows:

- first rotation round the third axis, $\underset{\sim}{R}_3(\gamma_1)$
- second rotation round the new second axis with the angle $(90°-\beta)$, $\underset{\sim}{R}_2(90°-\beta)$
- third rotation round the new third axis, $\underset{\sim}{R}_3(\gamma_2)$.

Table 1: The $\underset{\approx}{E}$ and $\underset{\approx}{F}$ reference frames

level	symbol	fundamental vector	Notation and name of the E-frame	3. base vector of the E-frame	2. base vector of the E-frame	1. base vector of the E-frame	Notation and name of the F-frame	1. base vector of the F-frame
0	'	Z	E' observational frame	$E_{3'} = \mathrm{norm}\ Z$ observational direction	$E_{2'} = \mathrm{norm}(-\Gamma x Z)$ (in the horizontal plane)	$E_{1'} = E_{2'} x E_{3'}$		
1	*	$-\Gamma$	E^* horizontal frame	$E_{3*} = \mathrm{norm}(-\Gamma)$ zenith	$E_{2*} = \mathrm{norm}(\Omega x(-\Gamma))$ east	$E_{1*} = E_{2*} x E_{3*}$ south	F^* theodolite frame	F_{1*}, direction "zero" of the azimuth circle of the theodolite
2	•	Ω	$E^{\bullet}$ equatorial frame (fixed in space)	$E_{3\bullet} = \mathrm{norm}\Omega$ north pole	$E_{2\bullet} = \mathrm{norm}(\psi x \Omega)$ autumn equinox	$E_{1\bullet} = E_{2\bullet} x E_{3\bullet}$	$F^{\bullet}$ equatorial frame (fixed in the earth)	$F_{1\bullet}$, in the Greenwich meridian
3	°	ψ	E° ecliptical frame (systematical)	$E_{3\circ} = \mathrm{norm}\psi$ northern pole of the ecliptic	$E_{2\circ} = \mathrm{norm}(V^4 x \psi)$ (in the average galaxy)	$E_{1\circ} = E_{2\circ} x E_{3\circ}$	F° ecliptical frame (conventional)	$F_{1\circ}$, vernal equinox
4			E^4 average galaxy frame	$E_{34} = \mathrm{norm}V^4$	$E_{24} = \mathrm{norm}(V^5 x V^4)$	$E_{14} = E_{24} x E_{34}$		
5	...							

In the matrix $\underline{R}_e$ the second rotation round the second axis would take place with the angle $\beta, \underline{R}_2(\beta)$.

These longitudinal and latitude angles are in detail:

$X_1^0 = X_*^{'} = A_s$ the azimuth of the observational direction;

$\Phi_1^0 = \Phi_*^{'} = B$ the vertical direction

$X_2^1 = X_{\bullet}^{*} = \theta_s$ the sidereal time

$\Phi_2^1 = \Phi_{\bullet}^{*} = \Phi$ the astronomical latitude.

For the transformation from a frame $\underset{\simeq}{E}^{i+1}$ to the frame $\underset{\simeq}{F}^i$ lying diagonally underneath, one needs additionally the orientation angle H^i (see Fig. 1):

$$\underset{\simeq}{F}^i = \underline{R}_E(X_{i+1}^i, \Phi_{i+1}^i, H^i)\underset{\simeq}{E}^{i+1} \tag{1-9}$$

For the transformation from a frame $\underset{\simeq}{F}^{i+1}$ *to the frame* $\underset{\simeq}{E}^i$ *lying diagonally* underneath, one needs the longitudinal angle Λ_{i+1}^i and the latitude angle Φ_{i+1}^i of the fundamental vector $\underset{\sim}{V}^i$ with regard to the frame $\underset{\simeq}{F}^{i+1}$:

$$\underset{\simeq}{E}^i = \underline{R}_E(\Lambda_{i+1}^i, \Phi_{i+1}^i)\underset{\simeq}{F}^{i+1} . \tag{1-10}$$

The latitude angles are the same as above, the longitudinal angles are in detail:

$\Lambda_1^0 = \Lambda_*^{'} = T_s$ the horizontal direction of the observation direction, systematically measured counter-clockwise, conventionally in the clockwise direction, $T_s = -T_c$;

$\Lambda_2^1 = \Lambda_{\bullet}^{*} = \Lambda$ the astronomical longitude;

$\Lambda_3^2 = \Lambda_{o}^{\bullet} = 90°$

and

$\Phi_3^2 = \Phi_{o}^{\bullet} = 90°-\varepsilon$ the orthogonal complement to the inclination of the ecliptic.

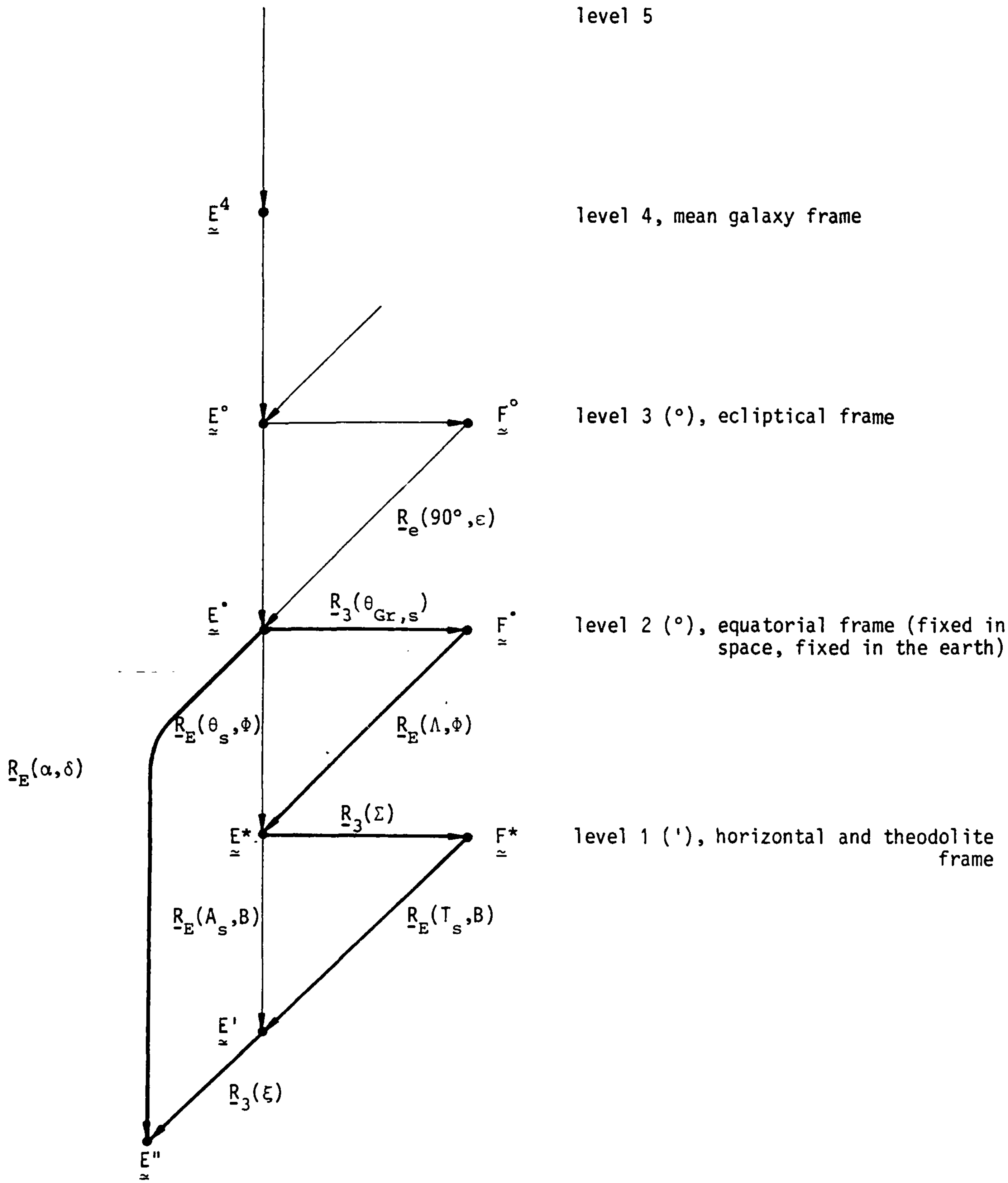


Fig. 1 : Commutative diagram with reference frames in geodetic astronomy

For the transformation from a frame $\underset{\approx}{F}^{i+1}$ to the underlying frame $\underset{\approx}{F}^{i}$, one needs additionally the orientation angle H^i:

$$\underset{\approx}{F}^{i} = \underset{-}{R}_E(\Lambda^i_{i+1}, \Phi^i_{i+1}, H^i)\underset{\approx}{F}^{i+1} \tag{1-11}$$

The transformation in the opposite direction results from the respective transposed of the rotation matrix.

1.2 The commutative diagram and the fundamental equation of geodetic astronomy

In the rotations introduced up to this point the right ascension α and the declination δ are still missing; these describe the observation direction $\underset{\sim}{E}_{3'}$ to a star with regard to the space fixed equator frame, namely in the same way as A and B do this with regard to the horizontal frame. Indeed, the rotation $\underset{-}{R}_E(\alpha,\delta)\underset{\approx}{E}^{\bullet}$ does not lead to the frame $\underset{\approx}{E}'$; in $\underset{\approx}{E}'' = \underset{-}{R}_E(\alpha,\delta)\underset{\approx}{E}^{\bullet}$ the second base vector $\underset{\sim}{E}_{2''}$ lies in the equatorial plane, but in $\underset{\approx}{E}'$ the vector $\underset{\sim}{E}_{2'}$ lies in the horizontal plane (east). Of course $\underset{\approx}{E}'$ and $\underset{\approx}{E}''$ have the common third base vector, but they differ by a rotation round this vector with an angle ξ :

$$\underset{\approx}{E}'' = \underset{-}{R}_3(\xi)\underset{\approx}{E}' \quad . \tag{1-12}$$

A serial connection of several transformations is called a diagram in the algebra. If the transformations which have to be reversibly unequivocal, form a closed circle, this is called a commutative diagram. Then one is able to express a transformation by means of the others. Such a commutative diagram is presented in Fig. 1 by the lines which are thickly marked. For example the transformation $\underset{\approx}{E}^* \to \underset{\approx}{E}'$ can be expressed by

$$\underset{\approx}{E}' = \underset{-}{R}_E(T,B)\underset{-}{R}_3(\Sigma)\underset{\approx}{E}^* \tag{1-13}$$

or

$$\underset{\approx}{E}' = \underset{-}{R}_3^T(\xi)\underset{-}{R}_E(\alpha,\delta)\underset{-}{R}_3^T(\theta_{Gr})\underset{-}{R}_E^T(\Lambda,\Phi)\underset{\approx}{E}^* \quad . \tag{1-14}$$

As the representation of the triad $\underset{\approx}{E}'$ with regard to the triad $\underset{\approx}{E}^*$ is unequivocal, it follows that:

$$\underline{R}_E(T,B)\underline{R}_3(\Sigma)=\underline{R}_3^T(\xi)\underline{R}_E(\alpha,\delta)\underline{R}_3^T(\theta_{Gr})\underline{R}_E^T(\Lambda,\Phi) \qquad (1\text{-}15)$$

This equation reads with the right-hand side written in full length

$$\underline{R}_E(T,B)\underline{R}_3(\Sigma)=\underline{R}_3(-\xi)\underline{R}_2(90°-\delta)\underline{R}_3(\alpha)\underline{R}_3(-\theta_{Gr})\underline{R}_3(-\Lambda)\underline{R}_2(\Phi-90°) \qquad (1\text{-}16)$$

and can again be reduced to

$$\underline{R}_E(T+\Sigma,B)=\underline{R}_e^T(\xi,\delta-90°)\underline{R}_E^T(\theta_{Gr}+\Lambda-\alpha,\Phi) \qquad (1\text{-}17)$$

These are the desired *fundamental relations* between the parameters appearing in geodetic astronomy.

2. The observation equations of geodetic astronomy

The fundamental equation (1-17) consists as a matrix equation of nine separate equations, of which only three are independent of each other because of the property of orthonormality of the rotation matrices. These three independent equations represent *condition equations with unknowns* for every star (if T,B and θ_{Gr} are measured at one instant).

The matrices on the left and right-hand side of equation (1-17) read as follows when they are multiplied respectively:

$$\begin{bmatrix} \cos(\Sigma+T)\sin B & \sin(\Sigma+T)\sin B & -\cos B \\ -\sin(\Sigma+T) & \cos(\Sigma+T) & 0 \\ \cos(\Sigma+T)\cos B & \sin(\Sigma+T)\cos B & \sin B \end{bmatrix}$$

and

Column 1:

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\sin\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\sin\xi+\cos\Phi\cos\delta\cos\xi$

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\sin\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\cos\xi+\cos\Phi\cos\delta\sin\xi$

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta-\cos\Phi\sin\delta$

Column 2:

$-\sin(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\cos(\theta_{Gr}+\Lambda-\alpha)\sin\xi$

$-\sin(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\cos(\theta_{Gr}+\Lambda-\alpha)\cos\xi$

$-\sin(\theta_{Gr}+\Lambda-\alpha)\cos\delta$

Column 3:

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\cos\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\sin\xi-\sin\Phi\cos\delta\cos\xi$

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\cos\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\cos\xi-\sin\Phi\cos\delta\sin\xi$

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta+\sin\Phi\sin\delta$

In the right-hand matrix the elements in the third row are the shortest and at the same time the only ones which do not contain the angle ξ. Therefore, it is the obvious thing to do to select two equations from this row as independent equations. As a third equation one could take an element from another row of the matrices whereby the angle ξ , in which one is not actually interested, would indeed appear as an additional unknown. So one, therefore, dispenses with such an equation and there remain only two independent equations for one complete observation (T,B and θ_{Gr}) with the three unknowns Λ, Φ,Σ:

$$\sin B = \cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta+\sin\Phi\sin\delta \tag{2-1}$$

$$\sin(\Sigma+T)\cos B = -\sin(\theta_{Gr}+\Lambda-\alpha)\cos\delta \tag{2-2}$$

$$\cos(\Sigma+T)\cos B = \sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta-\cos\Phi\sin\delta \tag{2-3}$$

Equations (2-1) and (2-2) are independent of each other, equation (2-3) is dependent on them both. It will be used later only for the determination of approximate values. The appearing variables be summarized once more:

Λ	astronomical longitude
Φ	astronomical latitude
α	right ascension of the star
δ	declination of the star
$h = \theta_{Gr} + \Lambda - \alpha$	hour angle
Σ	orientation unknown of the instrument (theodolite)
T	horizontal direction; observed
$\Rightarrow A = \Sigma + T$	azimuth
$B = 90° - z$	vertical direction, angle between horizon and star; observed
θ_{Gr}	Greenwich apparent sidereal time; observed

2.1 Linearization and matrix representation

The equations (2-1) and (2-2) are now linearized by developing them into a Taylor progression (and stopping after the first order term):

$$\begin{aligned}\underline{\sin B_0}+\cos B_0\,\delta B \doteq\ & \underline{\cos\Phi_0\cos h_0\cos\delta+\sin\Phi_0\sin\delta}\\ & -\cos\Phi_0\sin h_0\cos\delta(\delta\theta_{Gr}+\delta\Lambda)\\ & +(-\sin\Phi_0\cos h_0\cos\delta+\cos\Phi_0\sin\delta)\delta\Phi\end{aligned} \tag{2-4}$$

$$\begin{aligned}&\underline{\sin(\Sigma_0+T_0)\cos B_0}+\cos A_0\cos B_0(\delta\Sigma+\delta T)-\sin A_0\sin B_0\,\delta B\\ &\qquad \doteq \underline{-\sin h_0\cos\delta}-\cos h_0\cos\delta(\delta\theta_{Gr}+\delta\Lambda)\end{aligned} \tag{2-5}$$

Equations (2-4) and (2-5) contain the terms of (2-1) and (2-2) taken at the point of developing (underlined). By assuming that there are given approximate values for the unknowns Λ,Φ and Σ (these are $\Lambda_0,\Phi_0,\Sigma_0$) one can calculate values for B and T, so that equations (2-1) and (2-2) (and (2-3) because of the ambiguity of sine and cosine) are satisfied for these approximate values. Therefore, in equations (2-4) and (2-5) the terms underlined, taken at the Taylor-point, compensate. So it follows from (2-4) and (2-5) if one in addition places equation (2-3) in (2-4):

$$-\cos B_0\,\delta B-\cos\Phi_0\sin h_0\cos\delta(\delta\theta_{Gr}+\delta\Lambda)-\cos A_0\cos B_0\,\delta\Phi \doteq 0 \tag{2-6}$$

$$-\cos A_0\cos B_0(\delta\Sigma+\delta T)+\sin A_0\sin B_0\,\delta B-\cos h_0\cos\delta(\delta\theta_{Gr}+\delta\Lambda) \doteq 0 \tag{2-7}$$

From these equations one obtains the onset for the conditional equations in respect of the adjustment problem by introducing the vector $\underline{\varepsilon}$ of inconsistency. δB, δT and $\delta\theta_{Gr}$ represent in this case the difference between the actual observations to *one* star at *one* instant and the approximate values which go into the coefficients of the conditional equations:

$$\left.\begin{aligned}\delta B &= B - B_0\\ \delta T &= T - T_0\\ \delta\theta_{Gr} &= \theta_{Gr} - \theta_{Gr0}\end{aligned}\right\}\quad \begin{aligned}&\text{constitute the } \textit{vector of observations}\ \underline{y} = \begin{bmatrix}-\delta T\\ -\delta B\\ -\delta\theta_{Gr}\end{bmatrix}\\ &\Rightarrow E\{\underline{y}\} = \underline{y} - \underline{\varepsilon}\ .\end{aligned}$$

This is taken into consideration in equations (2-6) and (2-7); these equations are now remodelled:

$$-\cos\Phi_0 \sin h_0 \cos\delta\cdot\delta\Lambda-\cos A_0\cos B_0\cdot\delta\Phi-\cos B_0\cdot\varepsilon_B-\cos\Phi_0\sin h_0\cos\delta\cdot\varepsilon_{\theta Gr} = \\ = +\cos B_0\cdot\delta B+\cos\Phi_0\sin h_0\cos\delta\cdot\delta\theta_{Gr} \tag{2-8}$$

$$-\cos h_0\cos\delta\cdot\delta\Lambda-\cos A_0\cos B_0\cdot\delta\Sigma-\cos A_0\cos B_0\cdot\varepsilon_T+\sin A_0\sin B_0\cdot\varepsilon_B - \\ -\cos h_0\cos\delta\cdot\varepsilon_{\theta Gr} = \cos A_0\cos B_0\cdot\delta T-\sin A_0\sin B_0\cdot\delta B+\cos h_0\cos\delta\cdot\delta\theta_{Gr} \tag{2-9}$$

If observations are carried out in respect of several stars the general representation for the *Gauß-Helmert model* of conditional equations with unknowns is

$$A\underline{x} + B\underline{\varepsilon} = B\underline{y} - \underline{c} \tag{2-10}$$

Equation (2-10) runs in the case of consistent approximate values ($\underline{c} = 0$):

$$A\underline{x} + B\underline{\varepsilon} = B\underline{y} \quad : \quad B\cdot E\{\underline{y}\} = A\underline{x} \tag{2-11}$$

A and B are the coefficient matrices for all observations to all stars observed. In the case of three unknowns Λ,Φ,Σ and of observations to n stars A has the size 2n x 3 and B 2n x 3n and contain submatrices A_i and B_i for every triplet of observations to every star:

$$A = \begin{bmatrix} \underset{2x3}{A_1} \\ \hdashline \underset{2x3}{A_2} \\ \hdashline \underset{2x3}{A_3} \\ \hdashline \vdots \end{bmatrix} \tag{2-12}$$

$$B = \begin{bmatrix} \underset{2x3}{B_1} & & & \\ & \underset{2x3}{B_2} & & \\ & & \underset{2x3}{B_3} & \\ & & & \ddots \end{bmatrix} \tag{2-13}$$

The elements a_{ik} of the submatrices A_j and the elements $b_{\ell m}$ of the submatrices B_j respectively are now introduced.

In schematized form it, therefore, follows that:

$$a_{11}\delta\Lambda+a_{12}\delta\Phi+b_{12}\varepsilon_B+b_{13}\varepsilon_{\theta Gr} = -b_{12}\delta B-b_{13}\delta\theta_{Gr} \tag{2-14}$$

$$a_{21}\delta\Lambda+a_{23}\delta\Sigma+b_{21}\varepsilon_T+b_{22}\varepsilon_B+b_{23}\varepsilon_{\theta Gr} = -b_{21}\delta T-b_{22}\delta B-b_{23}\delta\theta_{Gr} \tag{2-15}$$

In matrix form this then becomes:

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \end{bmatrix} \begin{bmatrix} \delta\Lambda \\ \delta\Phi \\ \delta\Sigma \end{bmatrix} + \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} \varepsilon_T \\ \varepsilon_B \\ \varepsilon_{\theta_{Gr}} \end{bmatrix} = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} -\delta T \\ -\delta B \\ -\delta\theta_{Gr} \end{bmatrix} \tag{2-16}$$

$$A_j\underline{x}_j \qquad B_j\underline{\varepsilon}_j \qquad = \qquad B_j\underline{y}_j \tag{2-17}$$

No summation over the same indices!

Hence A_j and B_j are

$$A_j = \begin{bmatrix} -\cos\Phi_0\sin h_0\cos\delta & -\sin\Phi_0\cos h_0\cos\delta+\cos\Phi_0\sin\delta & 0 \\ -\cos h_0\cos\delta & 0 & -\cos A_0\cos B_0 \end{bmatrix} \tag{2-18}$$

$$B_j = \begin{bmatrix} 0 & -\cos B_0 & -\cos\Phi_0\sin h_0\cos\delta \\ -\cos A_0\cos B_0 & \sin A_0\sin B_0 & -\cos h_0\cos\delta \end{bmatrix} \tag{2-19}$$

One perceives that $a_{11} = b_{13}$, $a_{21} = b_{23}$, $a_{23} = b_{21}$.

2.2 The different types of observations

One can observe for *every* star

a) vertical direction and time of observation or
b) horizontal direction and time of observation or
c) horizontal direction, vertical direction and time of observation or
d) horizontal direction and vertical direction (without time).

Case c) corresponds to the construction in (2-14) to (2-19). The other cases are computed as follows. Thereby the starting point was taken as equation (2-6) = I and equation (2-7) = II, because one is only interested in the respective matrices A_j and B_j.

2.2.1 Case a): Vertical direction and time

For every pair of observables (that is the vertical direction B and the sidereal time θ_{Gr}) one obtains one conditional equation of type I:

$$\sin B = \cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta+\sin\Phi\sin\delta \tag{2-20}$$

which gives in linearized form

$$-\cos B_0\delta B-\cos\Phi_0\sin h_0\cos\delta(\delta\theta_{Gr}+\delta\Lambda)+(-\sin\Phi_0\cos h_0\cos\delta+\cos\Phi_0\sin\delta)\delta\Phi = 0\ . \tag{2-21}$$

Then the matrices read

$$B_j = [b_{12} \quad b_{13}] = [-\cos B \quad -\cos\Phi_0\sin(\theta_{Gr_0}+\Lambda_0-\alpha)\cos\delta] \tag{2-22}$$

$$\begin{aligned} A_j &= [a_{11} \quad a_{12}] = \\ &= [-\cos\Phi_0\sin(\theta_{Gr_0}+\Lambda_0-\alpha)\cos\delta \quad -\sin\Phi_0\cos(\theta_{Gr_0}+\Lambda_0-\alpha)\cos\delta+\cos\Phi_0\sin\delta] \end{aligned} \tag{2-23}$$

=>

$$[b_{12} \quad b_{13}] \begin{bmatrix} \delta B \\ \delta\theta_{Gr} \end{bmatrix} + [a_{11} \quad a_{12}] \begin{bmatrix} \delta\Lambda \\ \delta\Phi \end{bmatrix} = 0 \tag{2-24}$$

For n pairs of observations one obtains the nx2 matrix A and the nx2n matrix B.

2.2.2 Case b): Horizontal direction and time

δB must be eliminated in II. For that purpose equation I is solved for δB:

$$\delta B = -\frac{b_{13}}{b_{12}}\,\delta\theta_{Gr} - \frac{a_{11}}{b_{12}}\,\delta\Lambda - \frac{a_{12}}{b_{12}}\,\delta\Phi \tag{2-25}$$

and this is then inserted into II:

$$b_{21}\delta T+(b_{23}-b_{22}\frac{b_{13}}{b_{12}})\delta\theta_{Gr}+(a_{21}-b_{22}\frac{a_{11}}{b_{12}})\delta\Lambda-b_{22}\frac{a_{12}}{b_{12}}\delta\Phi+a_{23}\delta\Sigma = 0 \tag{2-26}$$

Thereby B_0 must be calculated from equation I.

Then the matrices read:

$$\begin{aligned} B_j &= [b_{21} \quad b_{23} - b_{22}\frac{b_{13}}{b_{12}}] \\ &= [-\cos A_0 \cos B_0 \quad -\cos h_0 \cos\delta - \sin A_0 \tan B_0 \cos\Phi_0 \sin h_0 \cos\delta] \end{aligned} \tag{2-27}$$

$$\begin{aligned} A_j &= [a_{21}-b_{22}\frac{a_{11}}{b_{12}} \quad -b_{22}\frac{a_{12}}{b_{12}} \quad a_{23}] \\ &= [-\cos h_0 \cos\delta - \sin A_0 \tan B_0 \cos\Phi_0 \sin h_0 \cos\delta \quad -\sin A_0 \tan B_0 (\sin\Phi_0 \cos h_0 \cos\delta - \cos\Phi_0 \sin\delta) \\ &\qquad -\cos A_0 \cos B_0] \end{aligned} \tag{2-28}$$

$$[b_{21} \quad b_{23} - b_{22}\frac{b_{13}}{b_{12}}] \begin{bmatrix} \delta T \\ \\ \delta\theta_{Gr} \end{bmatrix} + [a_{21}-b_{22}\frac{a_{11}}{b_{12}} \quad -b_{22}\frac{a_{12}}{b_{12}} \quad a_{23}] \begin{bmatrix} \delta\Lambda \\ \delta\Phi \\ \delta\Sigma \end{bmatrix} = 0 \tag{2-29}$$

For n pairs of observations one obtains the nx3 matrix A and the nx2n matrix B.

2.2.3 Case c): Horizontal direction, vertical direction and time

Case c) is represented in equations (2-14) to (2-19).

2.2.4 Case d): Horizontal direction and vertical direction

$\delta\theta_{Gr}$ must be eliminated in II. For that purpose equation I is solved for $\delta\theta_{Gr}$:

$$\delta\theta_{Gr} = -\frac{b_{12}}{b_{13}}\delta B - \frac{a_{11}}{b_{13}}\delta\Lambda - \frac{a_{12}}{b_{13}}\delta\Phi \tag{2-30}$$

and then this is inserted into II:

$$b_{21}\delta T + \left(b_{22} - \frac{b_{12}}{b_{13}} b_{23}\right)\delta B + \left(a_{21} - \frac{a_{11}}{b_{13}} b_{23}\right)\delta\Lambda - \frac{a_{12}}{b_{13}} b_{23}\delta\Phi + a_{23}\delta\Sigma = 0 \tag{2-31}$$

As stated above $a_{11} = b_{13}$, $a_{21} = b_{23}$.

This yields

$$a_{21} - \frac{a_{11}}{b_{13}} b_{23} = 0! \tag{2-32}$$

From this one obtains the final conditional equation

$$b_{21}\delta T + \left(b_{22} - \frac{b_{12}}{b_{13}} b_{23}\right)\delta B - \frac{a_{12}}{b_{13}} b_{23}\delta\Phi + a_{23}\delta\Sigma = 0 \tag{2-33}$$

Hence it follows that Λ cannot be estimated. The reference of the Greenwich meridian for θ_{Gr} and as a result the longitude Λ is in principle chosen arbitrarily so that Λ is not definable without time θ_{Gr}.

The matrices read

$$B_j = [b_{21} \quad b_{22} - \frac{b_{12}}{b_{13}} b_{23}]$$

$$= [-\cos A_0 \cos B_0 \quad \sin A_0 \sin B_0 + \frac{\cos B_0}{\cos\Phi_0 \tan h_0}]$$

$$A_j = [- \frac{a_{12}}{b_{13}} b_{23} \quad a_{23}]$$

$$= [\tan\Phi_0 \cos h_0 \coth_0 \cos\delta - \sin\delta \coth_0 \quad -\cos A_0 \cos B_0]$$

For n pairs of observations one obtains the nx2 matrix A and the nx2n matrix B.

3. Numerical studies

3.1 The adjustment model

As shown in chapter 2 the observations and unknowns are connected with each other in linearized form according to equation (2-10) or, in the case of consistent parameters, according to equation (2-11). This leads to the *Gauß-Helmert model* of condition equations with unknowns:

$$A\underline{x} + B\underline{\varepsilon} = B\underline{y} \tag{2-11}$$

In the realized simulation calculations one is actually not interested in the estimation of the unknowns $\underline{x}$, but only in the *accuracy* with which the unknowns can be estimated dependent upon the actual configuration of the stars. Thus one is only interested in the variance-covariance matrix of the unknowns. To determine this matrix the coefficient matrices for the present case of observations are first of all prepared: the parameter matrix A and the condition matrix B. Besides this the variance-covariance matrix Σ, i.e. the dispersion matrix $D(y)= \Sigma$ of the observation vector $\underline{y}$, has to be stipulated which is in general assumed to be a diagonal matrix.

With this the normal equation matrix N can be calculated:

$$N = B\Sigma B^T \tag{3-1}$$

Finally the variance-covariance matrix $Q_{\underline{\underline{xx}}}$ of the unknowns follows from that as

$$Q_{\underline{\underline{xx}}} = [A^T N^{-1} A]^{-1} = [A^T (B\Sigma B^T)^{-1} A]^{-1} \tag{3-2}$$

The square roots of the diagonal elements of $Q_{\underline{\underline{xx}}}$ are now the accuracies required with which the unknowns can be determined.

3.2 Simulation calculations

Simulation calculations have been carried out for the four different possible cases of observations of stars. Thereby the coordinates right ascension α and declination δ of fictitious stars were determined in dependence on the sidereal time in such a way that they approximately lay in defined directions as seen from the observation point. In accordance with this the simulation calculations were carried out, whereby the results in the diagrams are linked to the respective star configu-

rations as follows:

- ———————— the stars which are observed here are positioned approximately in the meridian of the observation point, both in the South as well as in the North (azimuth ≅ 0° and 180° respectively);
- - - - - - - - - - - the stars are positioned approximately in the first vertical; this is the great circle through the directions East and West (azimuth ≅ 90° and 270° respectively);
- -·-·-·-·-·-·- the stars have the hour angle $h \cong 6^h$ and $h \cong 18^h$ respectively, i.e. considered in the equatorial frame they have the angular distance ±90° from the meridian of the observation point measured along a parallel of latitude;
- -··-··-··-··- this line contains stars which are distributed over the whole firmament: meridian and first vertical;
- .... the stars of this set of observations are located partially in the meridian and partially they have the hour angle 6^h and 18^h respectively.

In the illustrations the mean errors (or standard deviations) of the unknowns astronomical longitude Λ, astronomical latitude Φ and orientation unknown of the theodolite Σ (as determinable in the different cases), which are to be expected, are drawn in dependence upon different parameters. For the four different cases of observations the accuracies in the determination of the unknowns are given first of all in dependence upon the number of the observed stars and subsequently in dependence upon the accuracy of the observations (horizontal and vertical direction or time measurement); here the calculations were carried out with ten observed stars.

For the calculation of the accuracy of the unknowns with the free parameter "number of stars" the accuracy of the observations has been supposed as:

horizontal direction : $\sigma_T = 1''$

vertical direction : $\sigma_{B'} = 1''$

time measurement : $\sigma_\theta = 0.1\text{sec}$

The accuracy of measuring the vertical direction $\sigma_{B'}$ is completed by the accuracy of the determination of the refraction, which can be calculated in dependence upon the vertical direction according to the equations in Appendix A.1. These results

differ, however, only by a few hundredths of a second of arc from those, which would follow when one neglects the inaccuracy of the refraction. In some illustrations the ordinate axis is drawn as a broken line. This means that the scale in the upper part of the diagram does not coincide with the scale in the lower part. The corresponding values are explicitly given.

The results shown in the different illustrations are strictly speaking valid only for the "observations" which have been supposed here. They are certainly dependent upon the constellation of the stars in the respective group of the observed stars. The tendency of the results will be correct anyway.

3.2.1 <u>Case a): Vertical direction and time</u>

Of course only accuracies for longitude and latitude could be calculated here because no horizontal directions were measured.

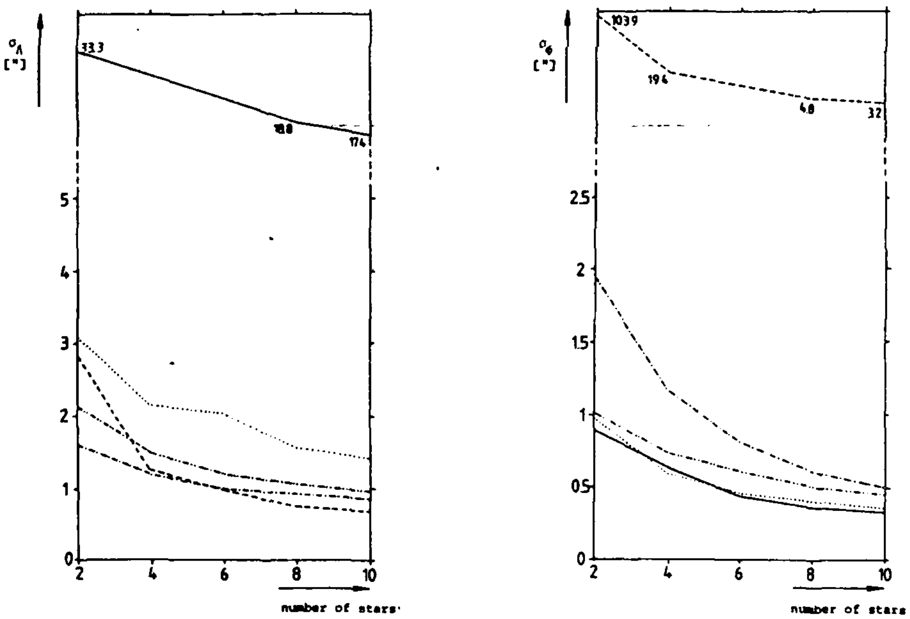


Fig. 2 and Fig. 3: Accuracies in dependence on the number of stars

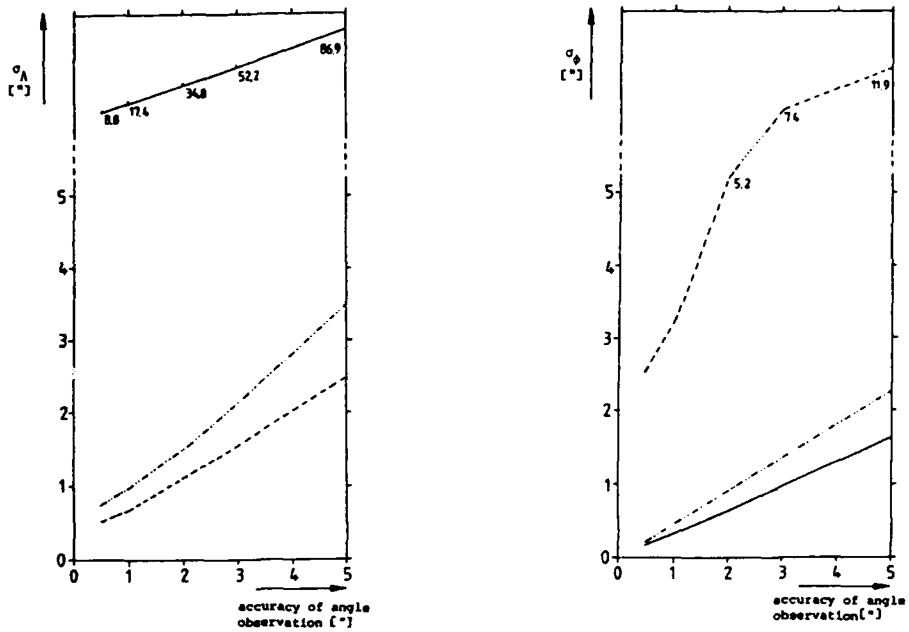


Fig. 4 and Fig. 5: Accuracies in dependence upon the accuracy of the measured angles (10 stars, σ_θ = 0.1 sec)

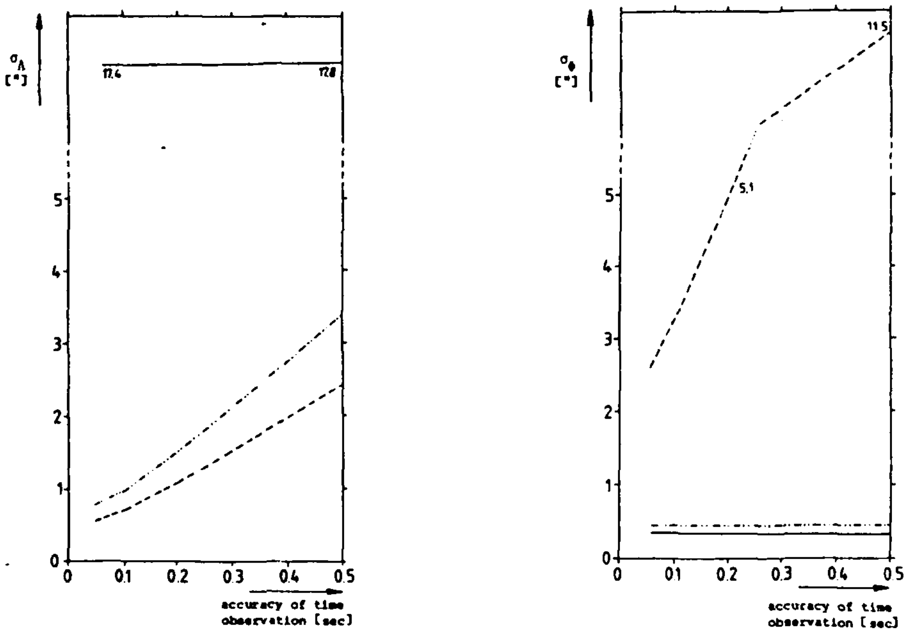
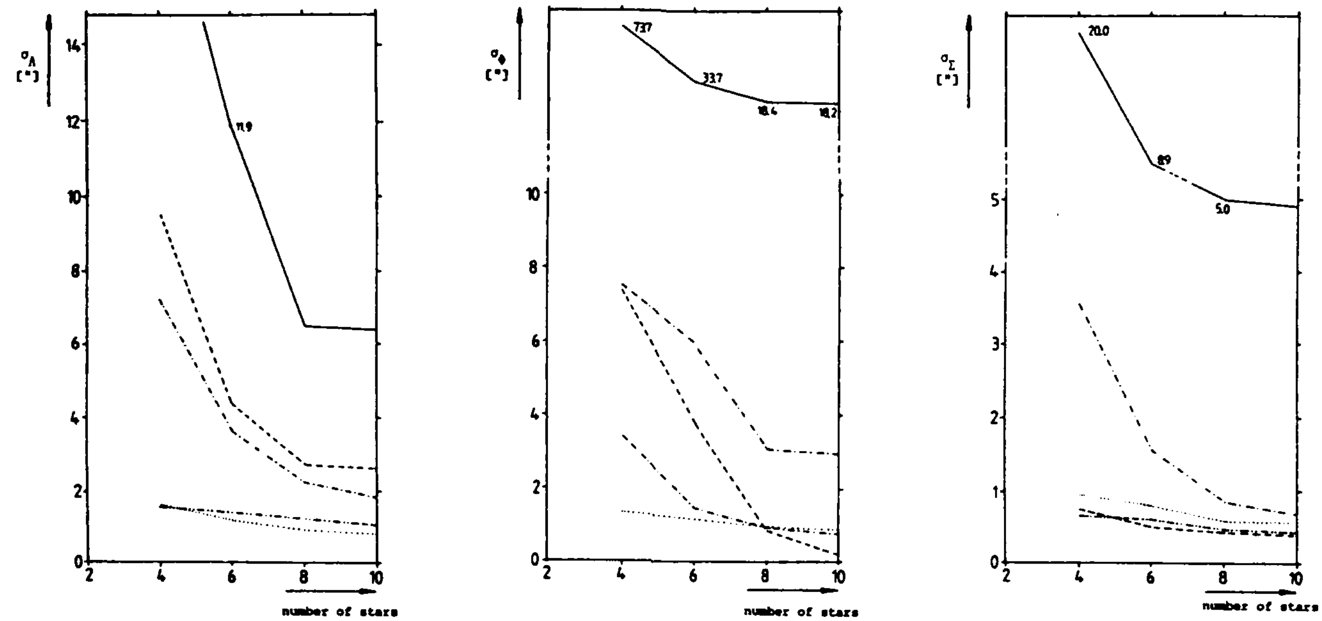


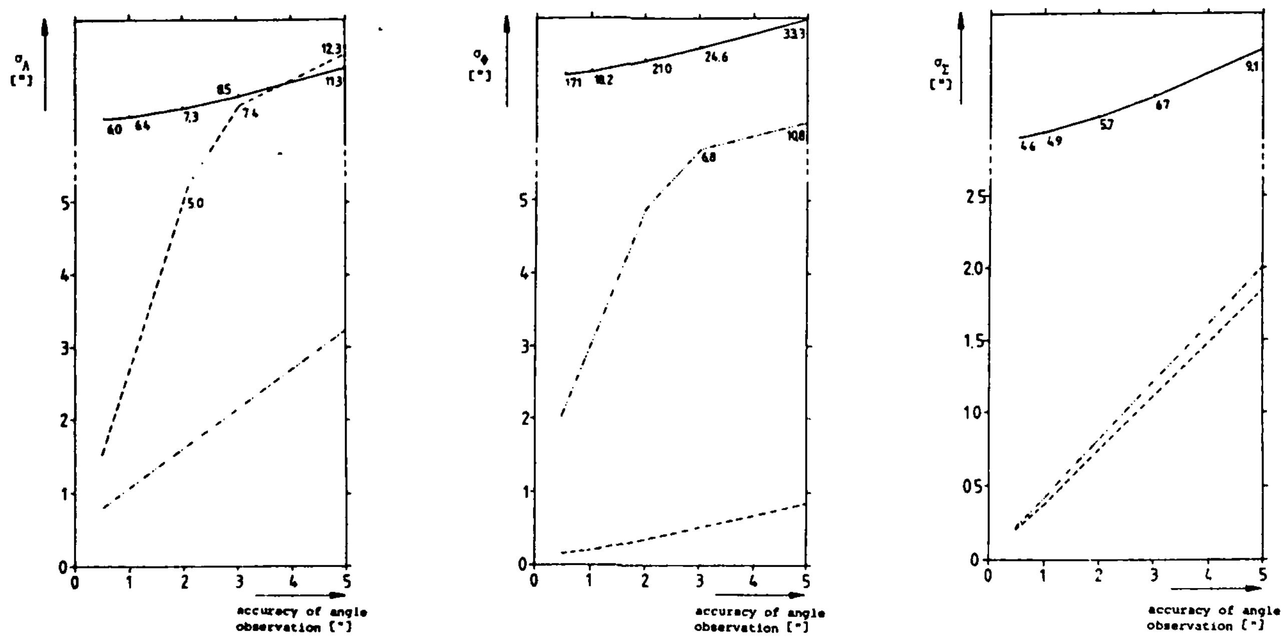
Fig. 6 and Fig. 7: Accuracies in dependence upon the accuracy of the time measurements (10 stars, $\sigma_{B'}$ = 1")

3.2.2 Case b): Horizontal direction and time

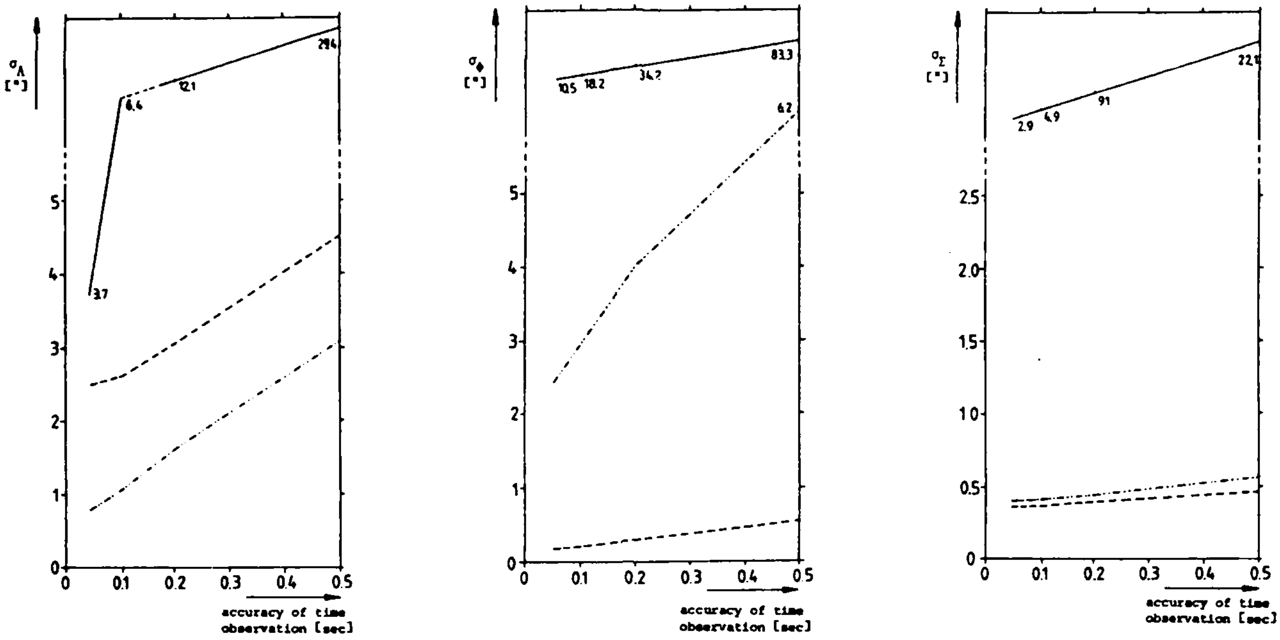
In the figures 8 to 10 the curves do not already begin with the observation of two stars because there is only one condition equation available for every star; thus at least three stars have to be observed.



Figures 8,9 and 10: Accuracies in dependence upon the number of stars



Figures 11,12 and 13: Accuracies in dependence upon the accuracy of the measured angles (10 stars, σ_θ = 0.1 sec)

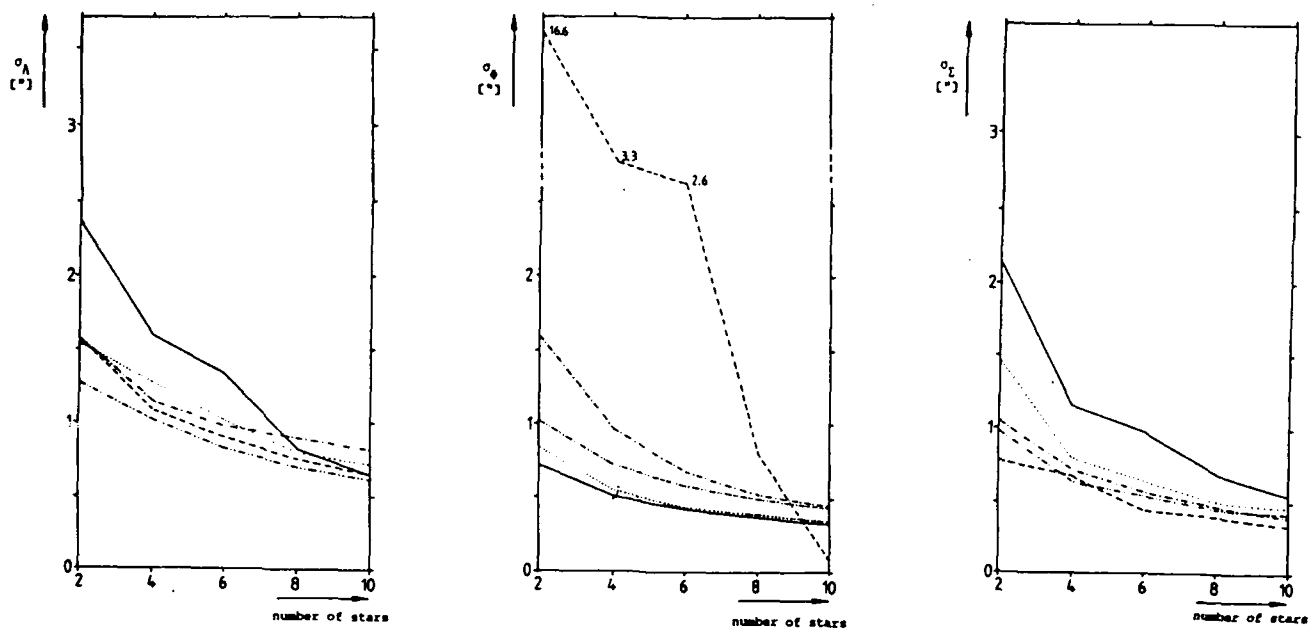


Figures 14,15 and 16: Accuracies in dependence upon the accuracy of the time measurements (10 stars, σ_T = 1")

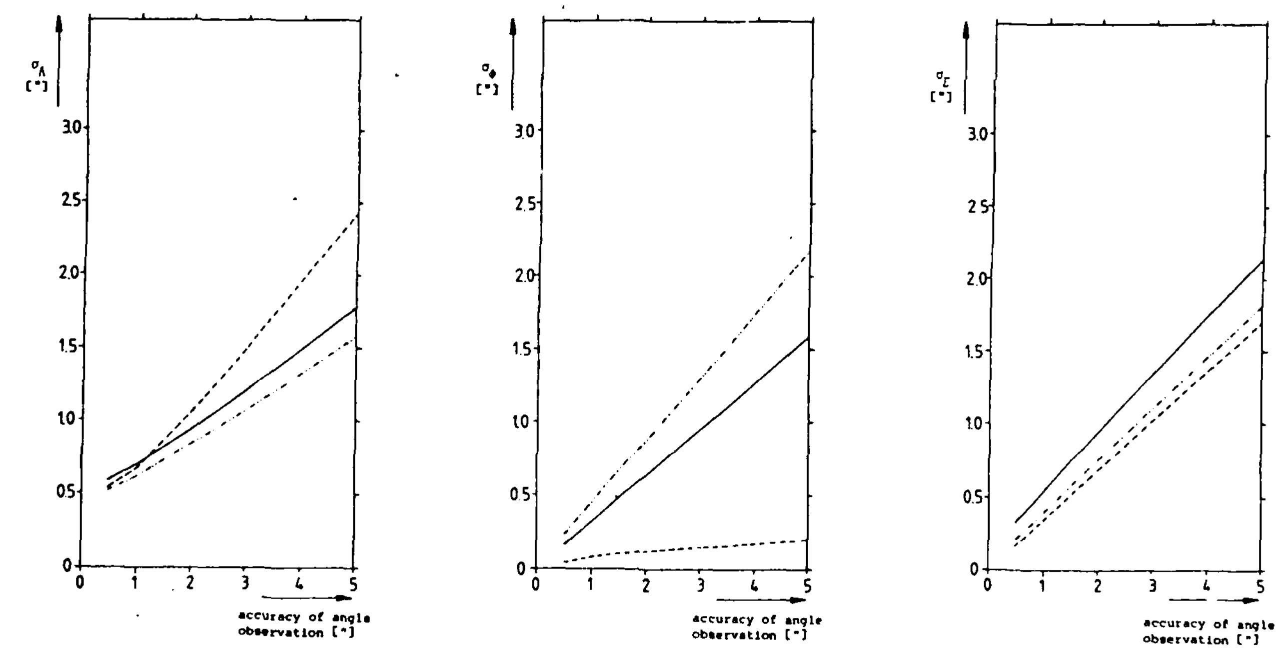
3.2.3 Case c): Horizontal direction, vertical direction and time

Figure 24 shows that in this case an inaccurate time measurement has hardly any effect on the determination of the astronomical latitude (if a sufficient number of stars have been observed).

Please see following page for figures 17 - 19.



Figures 17,18 and 19: Accuracies in dependence upon the number of stars



Figures 20, 21 and 22: Accuracies in dependence upon the accuracy of the measured angles (10 stars, σ_θ = 0.1 sec)

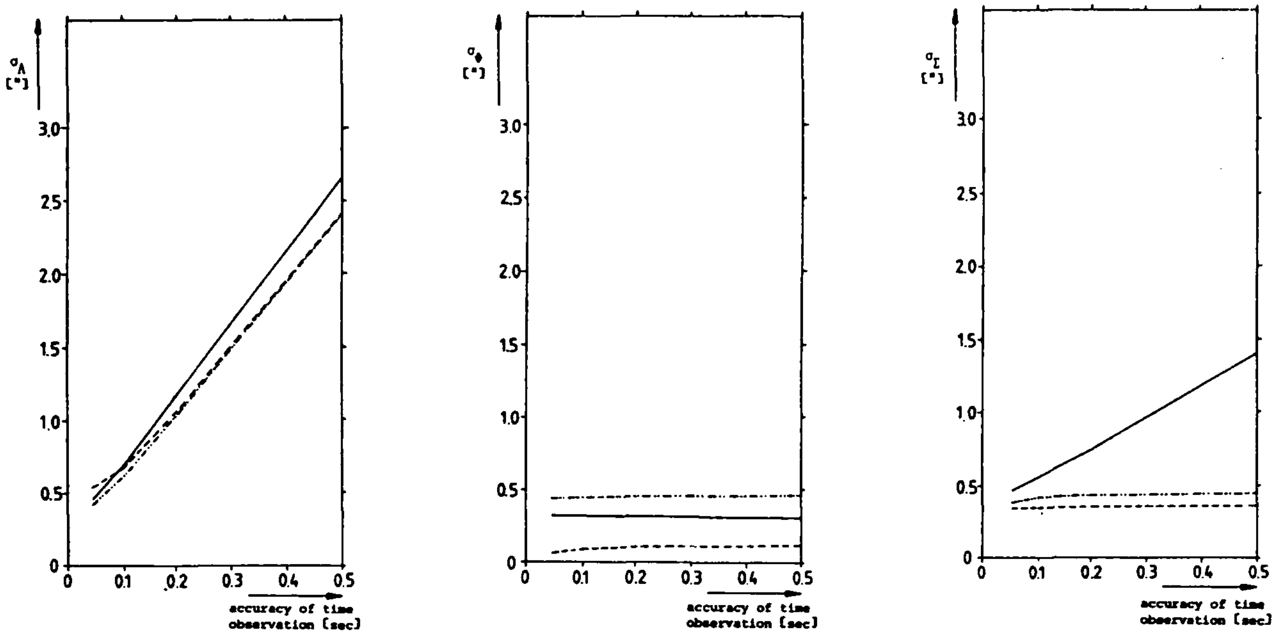


Fig. 23,24 and 25: Accuracies in dependence upon the accuracy of the time measurements (10 stars, $\sigma_T = \sigma_{B'} = 1''$)

3.2.4 Case d): Horizontal direction and vertical direction

As shown in 2.2.4 the astronomical longitude cannot be determined without measuring the time; thus the accuracy of the longitude cannot, of course, be specified either.

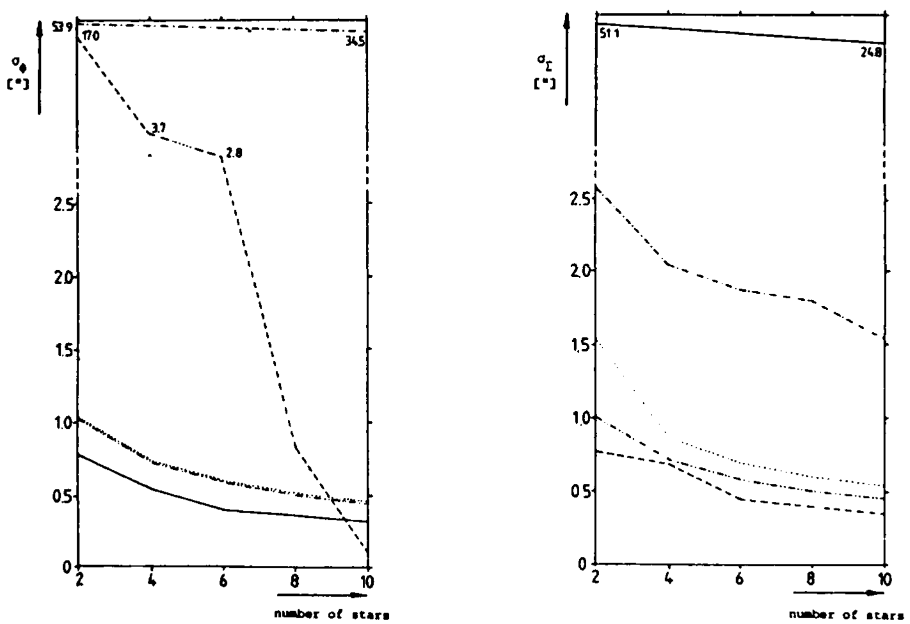


Fig. 26 and Fig. 27: Accuracies in dependence upon the number of stars

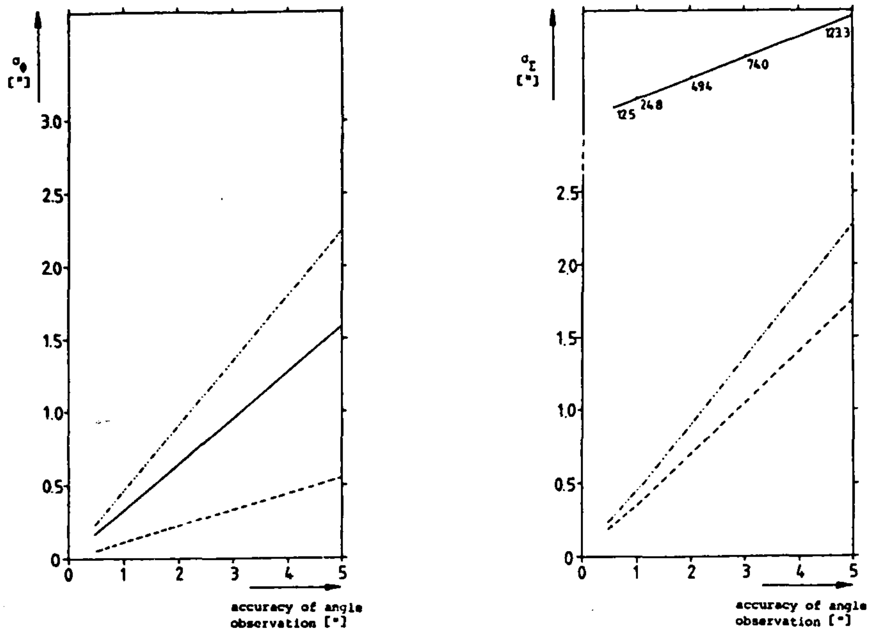


Fig. 28 and Fig. 29: Accuracies in dependence upon the accuracy of the measured angles (10 stars)

3.3 Computer programmes

Two computer programmes have been developed in FORTRAN 77, which are implemented in a PDP11/23+ computer. The first programme APPROX simulates the observation of a star. After the input of approximate values for the astronomical longitude and latitude of the observation point, the orientation unknown, right ascension and declination of a star and the sidereal time the programme calculates consistent values for horizontal and vertical direction with the equations (2-1) to (2-3) and writes the complete data set for every star onto a data file APPVAL. By choosing the respective parameters the observation directions required (see 3.2) can be computed.

The second programme ASTRO calculates the accuracies with which the unknowns can be estimated. In order to be able to do this it needs two data files as the input:

- the first field contains the chosen data sets of the observed stars consisting of the observations (according to case a) to d)) and the coordinates of the star, right ascension and declination (the value in the cases a), b) and d), which is not an "observation", i.e. horizontal direction, vertical direction or time, completes the data set as an approximate value);

- the second file contains three approximate values for the unknowns Λ,Φ and Σ.

The values have to be written onto the file in the format deg(or h)'(or min)" (or sec), for example 256 17 23.4 or 17 12 49.8. In Appendix A.2 the first programme APPROX and the second programme ASTROC for case c) are listed.

APPENDIX

A.1 The reduction of the observed vertical direction owing to refraction

A.1.1 The reduction formula

The vertical direction B to a star is influenced by the astronomical refraction. Therefore the measured vertical direction B' must be corrected by the value of the influence of refraction:

$$B = B' - R \tag{A-1}$$

R is the refraction, which is calculated by taking as a basis an atmospheric model and using temperature and atmospheric pressure measurements. The first step for calculation is the determination of the normal refraction R_0 which is valid for the following conditions at the observation point:

atmospheric pressure	p_0 = 760Torr = 1013,25 hpa
temperature	t_0 = 0°C
vapour pressure	e_0 = 6Torr

According to K. Ramsayer [Handbuch der Vermessungskunde, Band IIa: Geodätische Astronomie, 1968, p. 115ff] the normal refraction can be calculated with high precision by the approximation

$$R_0 = 60'',1012 \cdot \cot B - 0'',06483 \cdot \cot^3 B \qquad (B>20°) \tag{A-2}$$

This equation is based on refraction indices of the atmosphere which are calculated for different altitudes with a mean distribution of atmospheric pressure and temperature, which by the way is in good agreement with the U.S. Standard Atmosphere 1962.

In the second step there follows the reduction to another atmospheric pressure p and to another temperature t at the observation point. The equation for this reads as follows:

$$R = R_0 \frac{p[hpa]}{1013,25hpa} \cdot \frac{273,15}{273,15+t} \tag{A-3}$$

and with the abbreviations

$$G = \frac{p[hpa]}{1013,25hpa} \quad , \quad K = \frac{273,15}{273,15+t} \tag{A-4}$$

is

$$R = R_0 \cdot G \cdot K \tag{A-5}$$

The reduction to another vapour pressure e is calculated as follows:

$$dR = -0'',011(e-6) \cdot K \cdot \cot B$$

or in consideration of equation (A-2)

$$dR = -0,00018 \cdot R_0 \cdot K \cdot (e-6) \tag{A-6}$$

This reduction can generally be neglected.

A.1.2 The accuracy of the reduced vertical direction

The accuracy of the vertical direction depends firstly on the measuring accuracy and secondly on the accurcy of the calculation of the refraction. This problem will now be studied.

A.1.2.1 Deviations from the normal state of the atmosphere

According to K. Ramsayer the equation of refraction is valid:

$$R_0 = \rho(n_0-1)\cdot\cot B - \cot B\cdot\frac{\rho}{a}\cdot\int_{n_0}^{n_1} h\cdot d\left(\frac{n_0}{n}\right) + \frac{1}{2}\cdot\rho\cdot(n_0-1)^2\cdot\cot^3 B$$

$$-\cot^3 B\cdot\frac{\rho}{a}\cdot\int_{n_0}^{n_1} h\cdot d\left(\frac{n_0}{n}\right) \tag{A-7}$$

with

$\rho \quad = \quad (180\cdot3600)/\pi \quad ['']$

n_0 = refraction index at the observation point

a = radius of the osculating sphere at the observation point ("earth radius")

n = refraction index of a stratum of the atmosphere in the altitude h.

In the case of the mean state of the atmosphere equation (A-7) yields with p_0 = 760,3Torr, t_0 = 9,4°C, e_0 = 4,8Torr at the observation point the refractional value as follows:

$$R_0 = 58'',282\cdot\cot B-0'',076\cdot\cot B+0'',0082\cdot\cot^3 B-0'',0762\cdot\cot^3 B\ . \tag{A-8}$$

The comparison of equation (A-7) and (A-8) proves that the refraction primarily depends on the refraction index at the observation point and that it is to a large extent independent of the change of the refraction index which occurs with the change of the altitude.

Supposing a horizontal and plane stratification of the atmosphere ($a\to\infty$) equation (A-7) yields

$$R_0 \approx \rho(n_0-1)\cdot\cot B+\frac{\rho}{2}(n_0-1)^2\cdot\cot^3 B \tag{A-9}$$

Equation (A-9) shows that in the case of this assumption ($a\to\infty$) the refraction is only dependent on the vertical direction and the refraction index at the observation point and is completely independent of the state of the atmosphere above it. The fact that the state of the atmosphere does have an effect on the refraction all the same (in accordance with the integral terms in equation (A-7)) is the consequence of the curvature of the optical strata of the atmosphere. The integral terms have the following value for a mean state of the atmosphere

$$0'',076\cdot\cot B \quad \text{and} \quad 0'',0762\cdot\cot^3 B\ ,$$

and for B>30° they reach an order of magnitude of only a few tenths of a second of arc. Hence it follows with the assumption that the optical strata are concentric spheres that even considerable deviations from the normal state of the atmosphere (for example caused by a temperature inversion in the lower strata) can change the refraction at the most by 0",1 to 0",2.

So an estimation for the loss of accuracy of the refraction because of the deviation from the normal state of the atmosphere might be

$$dR_1 \approx 0'',05 \cdot \cot B \qquad (B>20°) \qquad \text{(A-10)}$$

or for the expected mean error of the determination of refraction because of the deviation from the normal state of the atmosphere

$$m_1 = \pm 0'',05 \cdot \cot B \qquad (B>20°) \qquad \text{(A-11a)}$$

This is equal to the standard deviation:

$$\sigma_1 = 0'',05 \cdot \cot B \qquad (B>20°) \qquad \text{(A-11b)}$$

A.1.2.2 Inclination of the strata and deviation from the spherical form

Most of the refraction theories postulate that the optical strata, i.e. the strata with the same refraction index of the atmosphere, are concentric spheres which are perpendicular to the plumb-line at the observation point. In reality these assumptions are not exactly complied with. One has to rather reckon with an inclination of the strata and a deviation from their spherical form.

An estimation of this influence yields

$$\Delta R \sim 0'',084 \cdot h[km] \cdot \frac{dT}{ds} \left[\frac{°C}{km}\right] \cdot \mathrm{cosec}^2 B \qquad \text{(A-12)}$$

The altitude of adjustment h specifies up to which altitude one has to reckon with an inclination of the strata. $\frac{dT}{ds}$ is the horizontal temperature gradient.

From series of observations which Harzer ["Berechnung der Ablenkungen der Lichtstrahlen in der Atmosphäre der Erde auf rein meteorologisch-physikalischer Grundlage", Publ. der Sternwarte in Kiel XIII, Germany, 1922-24] carried out,the influence of the deviation of the mean real optical surfaces from a concentric spherical form results in 0",01 for B = 70° and 0",06 for B = 30° respectively.

A valuation for this error might therefore be

$$\sigma_2 = 0'',015 \cdot \mathrm{cosec}^2 B \qquad \text{(A-13)}$$

A.1.2.3 The influence of the real water vapour pressure

The formula for the normal refraction uses a water vapour pressure of $e_0 = 6$Torr. According to K. Ramsayer the influence of the real vapour pressure e is

$$\Delta R = 0''011(e_0 - e) \cdot \cot B \;. \tag{A-14}$$

e oscillates seasonably between 4Torr in January and 12Torr in August (in Germany!).

The influence thereby becomes

$$|\Delta R| \leq 0'',07 \cdot \cot B \tag{A-15}$$

Generally it can be neglected. The expected mean error or the standard deviation could be stipulated to

$$\sigma_3 = 0'',02 \cdot \cot B \tag{A-16}$$

Hereby one supposed that in general e does not vary much from e_0.

A.1.2.4 The influence of the wave-length of the starlight

The refraction is dependent on the wave-length of the light emitted by the stars. The change of the refraction varies between $+0'',05 \cdot \cot B$ (white fixed stars) and $-0'',23 \cdot \cot B$ (red fixed stars). Thus the colour correction amounts to some tenths of a second of arc for B<45° in the case of reddish yellow to red stars, and should, therefore, be allowed for when calculating the refraction. An estimation for the mean error is, therefore, not made.

A.1.2.5 The influence of errors in the measurements

Finally, the influence of an error in measuring atmospheric pressure and temperature must also be considered. Starting from equation (A-3) one obtains the following when assuming the values $R \approx 60'' \cdot \cot B$, p = 760Torr, T = 283°K:

$$dR \approx 0'',079 \cdot \cot B \cdot dp - 0'',212 \cdot \cot B \cdot dT \tag{A-17}$$

The change to the standard deviation yields with $\sigma_p = 0,1$Torr and $\sigma_T = 0,1$°C

$$\sigma_5 = \cot B \cdot \sqrt{0,0079^2 + 0,0212^2} \quad = \quad 0'',023 \cdot \cot B \tag{A-18}$$

So the influence of an error in pressure and temperature can in general be neglected.

A.1.2.6 Summary of the different errors

Summing up the proceeding influences on refraction this yields

$$\sigma_R = \sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_5^2}$$

$$= \sqrt{[(0'',05)^2+(0'',02)^2+(0'',023)^2]\cot^2 B+(0'',015)^2 \operatorname{cosec}^4 B}$$

and finally

$$\sigma_R = \sqrt{(0'',06\cdot\cot B)^2+(0'',015\cdot\operatorname{cosec}^2 B)^2} \quad . \qquad \text{(A-19)}$$

This error has to be added to the observational error σ_0 from which the standard deviation of the corrected vertical direction B results:

$$\sigma_B = \sqrt{\sigma_0^2 + \sigma_R^2} \qquad \text{(A-20)}$$

A.1.2.7 Comparison with series of observations

Im comparison with equation (A-19) the results of series of observations of meridian vertical directions lasting one and two years carried out by J. Bauschinger (1898) and L. Courvoisier (1904) are presented. Averaging the differences between calculated and observed vertical directions one obtains

$$\text{Bauschinger} : \quad \sigma \approx \sqrt{(0'',31)^2+(0'',0035\cdot\operatorname{cosec}^2 B)^2} \qquad \text{(A-21)}$$

$$\text{Courvoisier} : \quad \sigma \approx \sqrt{(0'',23)^2+(0'',032\cdot\operatorname{cosec}^2 B)^2} \qquad \text{(A-22)}$$

The first terms in the square root denote the influence of the observation errors. The second terms can be interpreted as the mean influence of the inclination of the optical strata and represent here the real error of the determination of the refraction. So the value in equation (A-13) assumed for the influence of the in-

clination of the optical strata lies between those values which were deducted from observations in equations (A-21) and (A-22). The parts of the errors as in equations (A-11) and (A-16) do not appear here since convenient conditions were probably present for the observations. The part of the error from equation (A-18) could be eliminated by observing diametric stars lying more or less in the same horizontal distance and by averaging these observations.

A.2 Programme listings

A.2.1 Programme APPROX

```
      PROGRAM APPROX
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL MM
      CHARACTER*1 DUMMY
      PI=4.*ATAN(1.)
      OPEN(1,FILE='APPVAL',STATUS='NEW')
    1 WRITE(5,100)
  100 FORMAT(5X,'astronomical longitude (deg '' '')')
      READ(5,*)G,BM,BS
      CALL OUT(G,BM,BS,AL,1)
      WRITE(5,101)
  101 FORMAT(5X,'astronomical latitude (deg '' '')')
      READ(5,*)G,BM,BS
      CALL OUT(G,BM,BS,AB,2)
      WRITE(5,102)
  102 FORMAT(5X,'orientation unknown (deg '' '')')
      READ(5,*)G,BM,BS
      CALL OUT(G,BM,BS,OU,3)
      WRITE(5,103)
  103 FORMAT(5X,'right ascension (h min sec)')
      READ(5,*)H,MM,SS
      CALL OUT(H,MM,SS,RA,4)
      WRITE(5,104)
  104 FORMAT(5X,'declination (deg '' '')')
      READ(5,*)G,BM,BS
      CALL OUT(G,BM,BS,D,5)
      WRITE(5,105)
  105 FORMAT(5X,'Greenwich sidereal time (GAST) (h min sec)')
      READ(5,*)H,MM,SS
      CALL OUT(H,MM,SS,TH,6)
      SINB=DCOS(AB)*DCOS(TH+AL-RA)*DCOS(D)+DSIN(AB)*DSIN(D)
      B=DASIN(SINB)
      SINAZ=-DSIN(TH+AL-RA)*DCOS(D)/DCOS(B)
      AZ1=DASIN(SINAZ)
      COSAZ=(DSIN(AB)*DCOS(TH+AL-RA)*DCOS(D)-DCOS(AB)*DSIN(D))
     + /DCOS(B)
      AZ2=DACOS(COSAZ)
      IF(SINAZ.GE.0..AND.COSAZ.GE.0.) AZ=AZ2
      IF(SINAZ.GE.0..AND.COSAZ.LT.0.) AZ=AZ2
      IF(SINAZ.LT.0..AND.COSAZ.LE.0.) AZ=2.*PI-AZ2
      IF(SINAZ.LT.0..AND.COSAZ.GT.0.) AZ=2.*PI-AZ2
      T=AZ-OU
      IF(T.LT.0.) T=T+2.*PI
C     Output to the terminal
      WRITE(5,'(///,5X,''longitude:'',F12.6,'' deg'')') AL*180./PI
      WRITE(5,'(5X,''latitude:'',F12.6,'' deg'')') AB*180./PI
```

```
      WRITE(5,'(5X,"orientation unknown:",F12.6," deg")')
     + OU*180./PI
      WRITE(5,'(5X,"right ascension:"F12.6," h")')RA*180./15./PI
      WRITE(5,'(5X,"declination:",F12.6," deg")') D*180./PI
      WRITE(5,'(5X,"sidereal time:",F12.6," h")') TH*180./15./PI
      WRITE(5,'(5X,"vertical direction:",F12.6," deg")')B*180./PI
      WRITE(5,'(5X,"horizontal direction:",F12.6," deg")')
     + T*180./PI
      WRITE(5,'(5X,"azimuth:",F12.6," deg")') AZ*180./PI
      WRITE(5,'(5X,"sin(azimuth):",F12.6)') SINAZ
      WRITE(5,'(5X,"cos(azimuth):",F12.6)') COSAZ
      B=B*180./PI
      T=T*180./PI
      AZ=AZ*180./PI
      IG=INT(B)
      IF(B.LT.0.) IG=-IG
      BM=ABS(B-IG)*60.
      IBM=INT(BM)
      BS=(BM-IBM)*60.
      WRITE(1,106)IG,IBM,BS
106   FORMAT(5X,'vertical direction:',I4,'deg',I3,'''',F5.1,'''')
      IG=INT(T)
      BM=(T-IG)*60.
      IBM=INT(BM)
      BS=(BM-IBM)*60.
      WRITE(1,107)IG,IBM,BS
107   FORMAT(5X,'horizontal direction:',I4,'deg',I3,'''',F5.1,'''')
      IG=INT(AZ)
      BM=(AZ-IG)*60.
      IBM=INT(BM)
      BS=(BM-IBM)*60.
      WRITE(1,108)IG,IBM,BS
108   FORMAT(5X,'azimuth:',I4,'deg',I3,'''',F5.1,'''')
      WRITE(1,'(5X,"sin(azimuth):",F12.6,5X,"cos(azimuth):",
     + F10.6)') SINAZ,COSAZ
      WRITE(5,'(5X,"Should a new data set be calculated (Y/NÜ ?
     + ")')
      READ(5,'(A1)') DUMMY
      IF(DUMMY.EQ.'Y'.OR.DUMMY.EQ.'y'.OR DUMMY.EQ.'1') GOTO 1
      STOP
      END
C
C
C
C
C
```

```
      SUBROUTINE OUT(G,BM,BS,RET,N)
      DOUBLE PRECISION G,BM,BS,RET,PI
C     Output to the data file APPVAL
      PI=4.0*ATAN(1.0)
      IG=INT(G)
      IBM=INT(BM)
      IF(N.GT.1) GOTO 10
      WRITE(1,200)IG,IBM,BS
  200 FORMAT(///,5X,'astronomical longitude:',I4,'deg',I3,'''',
     + F5.1,'"')
      RET=(G+BM/60.+BS/3600.)*PI/180.
      GOTO 60
   10 IF(N.GT.2) GOTO 20
      WRITE(1,201)IG,IBM,BS
  201 FORMAT(5X,'astronomical latitude:',I4,'deg',I3,'''',F5.1,'"')
      RET=(G+BM/60.+BS/3600.)*PI/180.
      GOTO 60
   20 IF(N.GT.3) GOTO 30
      WRITE(1,202)IG,IBM,BS
  202 FORMAT(5X,'orientation unknown:',I4,'deg',I3,'''',F5.1,'"')
      RET=(G+BM/60.+BS/3600.)*PI/180.
      GOTO 60
   30 IF(N.GT.4) GOTO 40
      WRITE(1,203)IG,IBM,BS
  203 FORMAT(5X,'right ascension:',I4,'h',I3,'min',F5.1,'sec')
      RET=(G+BM/60.+BS/3600.)*PI/180.*15.
      GOTO 60
   40 IF(N.GT.5) GOTO 50
      WRITE(1,204)IG,IBM,BS
  204 FORMAT(5X,'declination:',I4,'deg',I3,'''',F5.1,'"')
      RET=(G+BM/60.+BS/3600.)*PI/180.
      GOTO 60
   50 CONTINUE
      WRITE(1,205)IG,IBM,BS
  205 FORMAT(5X,'Greenwich sidereal time:',I4,'h',I3,'min',F5.1,'sec')
      RET=(G+BM/60.+BS/3600.)*15.*PI/180.
   60 RETURN
      END
```

A.2.2 Programme ASTROC

```
      PROGRAM ASTROC
      REAL MM
      PARAMETER (KK=10)
      DIMENSION SIGMA(3*KK),U(3),DISP(3,3)
      DIMENSION Y(KK,3),SIGY(3*KK,3*KK),A(2*KK,3),
     + B(2*KK,3*KK),BT(3*KK,2*KK),AT(3,2*KK),RN(2*KK,2*KK),
     + ATN(3,2*KK),HILF(2*KK,3*KK),NUM(2*KK),D(2*KK),DELTA(KK),
     + ALPHA(KK)
      CHARACTER FRAGE*1
      DATA RN/400*0.D0/
C
C
      CHARACTER *25 FNAME1,FNAME2
      LOGICAL LOG
C
C
C
      H=0.
      G=0.
      MM=0.
      BM=0.
      SS=0.
      BS=0.
      PI=4.*ATAN(1.)
      WRITE(5,'(//,5X,"name of the data file?")')
      READ(5,'(A25)')FNAME1
      WRITE(5,'(//,5X,"name of the file with the approximation values",
     + " for the unknowns ?")')
      READ(5,'(A25)')FNAME2
      WRITE(5,'(//,5X,"accuracy of horizontal directions (") ?")')
      READ(5,*)BS
      GHR=BS
      CALL DEZIG(G,BM,BS,SIGMA(1))
      CALL RADI(SIGMA(1),PI)
      WRITE(5,'(//,5X,"accuracy of vertical directions (") ?")')
      READ(5,*)BS
      GHD=BS
      WRITE(5,'(//,5X,"consideration of the accuracy of refraction",
     + " (Y/N) ?")')
      READ(5,'(A1)')FRAGE
      WRITE(5,'(//,5X,"accuracy of time observations (sec) ?")')
      READ(5,*)SS
      GZ=SS
      CALL DEZIH(H,MM,SS,SIGMA(3))
      CALL RADI(SIGMA(3),PI)
C
```

```
C
C
      OPEN(1,FILE=FNAME1,ACCESS='SEQUENTIAL',STATUS='OLD')
      OPEN(2,FILE=FNAME2,ACCESS='SEQUENTIAL',STATUS='OLD')
      OPEN(3,FILE='OUTC.DAT',STATUS='NEW')
C
C
C
      READ(2,*)G,BM,BS
      IG=INT(G)
      IBM=INT(BM)
      CALL DEZIG(G,BM,BS,U(1))
      CALL RADI(U(1),PI)
      READ(2,*)G,BM,BS
      IG=INT(G)
      IBM=INT(BM)
      CALL DEZIG(G,BM,BS,U(2))
      CALL RADI(U(2),PI)
      READ(2,*)G,BM,BS
      IG=INT(G)
      IBM=INT(BM)
      CALL DEZIG(G,BM,BS,U(3))
      CALL RADI(U(3),PI)
C
C
C
C
C
      WRITE(3,700)
  700 FORMAT(///,5X,'observations',/,5x,'turn: horizontal ',
     + 'direction ','(deg-''-")',/,11x,'vertical direction ',
     + '(deg-''-")',/,11X,'sidereal time (GAST) (h-min-sec)',
     + /,11X,'and',/,11X,'right ascension (h-min-sec)',/,11X,
     + 'declination (deg-''-")')
      K=1
   25 READ(1,*,END=20)
      K=K+1
      GOTO 25
   20 K=K/5
      REWIND 1
C
C
      DO 10 J=1,K
      WRITE(3,'(/)')
      DO 30 I=1,2
      READ(1,*)G,BM,BS
      IG=INT(G)
      IBM=INT(BM)
      WRITE(3,800)IG,IBM,BS
  800 FORMAT(5X,I4,'deg',I3,'''',F5.1,'"')
      CALL DEZIG(G,BM,BS,Y(J,I))
      CALL RADI(Y(J,I),PI)
```

```
      IF(I.EQ.2) THEN
      BS=SQRT(GHD*GHD+(0.06/TAN(Y(J,I)))**2.+
     +(0.015/(SIN(Y(J,I)))**2.)**2.)
      IF(FRAGE.EQ.'N'.OR.FRAGE.EQ.'n'.OR.FRAGE.EQ.'0')THEN
      BS=GHD
      ENDIF
      CALL DEZIG(0.,0.,BS,SIGMA(3*(J-1)+2))
      CALL RADI(SIGMA(3*(J-1)+2),PI)
       ELSE
      SIGMA(3*(J-1)+1)=SIGMA(1)
      ENDIF
   30 CONTINUE
      READ(1,*)H,MM,SS
      IH=INT(H)
      IMM=INT(MM)
      WRITE(3,'(5X,I4,"h",I3,"min",F5.1,"sec")')IH,IMM,SS
      CALL DEZIH(H,MM,SS,Y(J,3))
      CALL RADI(Y(J,3),PI)
      SIGMA(3*(J-1)+3)=SIGMA(3)
      READ(1,*)H,MM,SS
      IH=INT(H)
      IMM=INT(MM)
      WRITE(3,'(5X,I4,"h",I3,"min",F5.1,"sec")')IH,IMM,SS
      CALL DEZIH(H,MM,SS,ALPHA(J))
      CALL RADI(ALPHA(J),PI)
      READ(1,*)G,BM,BS
      IG=INT(G)
      IBM=INT(BM)
      WRITE(3,800)IG,IBM,BS
      CALL DEZIG(G,BM,BS,DELTA(J))
      CALL RADI(DELTA(J),PI)
   10 CONTINUE
      WRITE(3,801)GHR
  801 FORMAT(/,5X,'accuracy of horizontal directions:',F6.2,
     +'''')
      WRITE(3,802)GHD
  802 FORMAT(5X,'accuracy of vertical directions (observations):'
     +,F6.2,'''')
      WRITE(3,803)GZ
  803 FORMAT(5X,'accuracy of time observations:',F6.2,'sec')
C
C
C
      DO 11 I=1,3*KK
      DO 11 J=1,3*KK
   11 SIGY(I,J)=0.
      DO 12 I=1,2*KK
      DO 12 J=1,3
   12 A(I,J)=0.
      DO 13 I=1,2*KK
      DO 13 J=1,3*KK
   13 B(I,J)=0.
```

```
      DO 14 I=1,3*K
   14 WRITE(3,*)SIGMA(I)*180./PI*3600.
      CALL SIGMAS(SIGY,KK,K,SIGMA)
C     SIGMA = vector of the standard deviations of the observations (in (rad))
C     SIGY  = matrix of the variances of the observations
      FAKTOR=SIGY(1,1)
      DO 32 I=1,3*K
      DO 32 J=1,3*K
      SIGY(I,J)=SIGY(I,J)/FAKTOR
   32 CONTINUE
      CALL AMAT(A,Y,U,KK,K,ALPHA,DELTA)
      DO 40 I=1,2*K
      WRITE(3,300)(A(I,M),M=1,3)
  300 FORMAT(5X,3F18.12)
   40 CONTINUE
      CALL BMAT(B,Y,U,KK,K,ALPHA,DELTA)
      I=0
      DO 41 J=1,2*K-1,2
      WRITE(3,300)(B(J,I*3+M),M=1,3)
      WRITE(3,300)(B(J+1,I*3+M),M=1,3)
      WRITE(3,'(/)')
   41 I=I+1
C
C
C
      CALL MATMUL(B,2*KK,2*K,3*KK,3*K,SIGY,3*KK,3*K,3*KK,3*K,
     + HILF,2*KK,2*K,3*KK,3*K)
      DO 50 I=1,2*K
      DO 50 J=1,3*K
   50 BT(J,I)=B(I,J)
      CALL MATMUL(HILF,2*KK,2*K,3*KK,3*K,BT,3*KK,3*K,2*KK,2*K,RN,
     + 2*KK,2*K,2*KK,2*K)
C     normal equations matrix
      WRITE(3,'(5X,"matrix B*SIGY*BT:")')
      DO 51 I=1,2*K
   51 WRITE(3,401)(RN(I,J),J=1,2*K)
      CALL INVER2(RN,2*K,NUM,LOG,D,2*KK)
      WRITE(3,'(5x,"determinant of (B*SIGY*BT):",
     + 3X,E18.10)')D(1)
      WRITE(3,'(5X,"matrix (B*SIGY*BT)^-1:")')
      DO 52 I=1,2*K
   52 WRITE(3,401)(RN(I,J),J=1,2*K)
  401 FORMAT(8E15.6)
      DO 60 I=1,2*K
      DO 60 J=1,3
   60 AT(J,I)=A(I,J)
      CALL MATMUL(AT,3,3,2*KK,2*K,RN,2*KK,2*K,2*KK,2*K,ATN,3,3,
     + 2*KK,2*K)
      CALL MATMUL(ATN,3,3,2*KK,2*K,A,2*KK,2*K,3,3,DISP,3,3,3,3)
      CALL INVER2(DISP,3,NUM,LOG,D,3)
      WRITE(3,'(5x,"determinant of (AT*((B*SIGY*BT)^-1)*A):",
     + 3X,E18.10)')D(1)
```

```
C
C
C
C     variance-covariance-matrix of the unknowns
      DO 70 I=1,3
   70 WRITE(3,500)(DISP(I,J)*FAKTOR,J=1,3)
  500 FORMAT(3F22.16)
      DO 90 I=1,3
      DISP(I,I)=SQRT(DISP(I,I))*180./PI*SQRT(FAKTOR)
      DISP(I,I)=DISP(I,I)*3600.
C    standard deviations of the unknowns (in seconds of arc)
   90 WRITE(3,600) DISP(I,I)
  600 FORMAT(5X,F12.4,' "')
      STOP
      END
C
C
C
C
C
      SUBROUTINE DEZIG(G,BM,BS,RET)
C    DEZIG transforms the input (degree,minute of arc,second of arc)
C         into a decimal value (degree)
      RET=G+BM/60.+BS/3600.
      RETURN
      END
C
C
C
      SUBROUTINE DEZIH(H,MM,SS,RET)
C    DEZIH transforms the input (hour,minute of arc,second of arc)
C         into a decimal value (degree)
      REAL MM
      RET=(H+MM/60.+SS/3600.)*15.
      RETURN
      END
C
C
C
      SUBROUTINE RADI(RET,PI)
C    RADI transforms from (degree) to (rad)
      RET=RET*PI/180.
      RETURN
      END
C
C
C
```

```
      SUBROUTINE SIGMAS(SIGY,KK,K,SIGMA)
C     SIGMAS constructs the variance-covariance-matrix of the observations
      DIMENSION SIGY(3*KK,3*KK),SIGMA(3*KK)
      DO 10 I=1,3*K
      SIGY(I,I)=SIGMA(I)*SIGMA(I)
   10 CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE AMAT(A,Y,U,KK,K,ALPHA,DELTA)
C     AMAT constructs the matrix A
      DIMENSION A(2*KK,3),Y(KK,3),U(3),ALPHA(KK),DELTA(KK)
      J=1
      DO 10 I=1,2*K-1,2
      A(I,1)=-COS(U(2))*SIN(Y((I+1)/2,3)+U(1)-ALPHA(J))*COS(DELTA(J))
      A(I,2)=-SIN(U(2))*COS(Y((I+1)/2,3)+U(1)-ALPHA(J))*COS(DELTA(J))
     + +COS(U(2))*SIN(DELTA(J))
      J=J+1
   10 CONTINUE
      J=1
      DO 20 I=2,2*K,2
      A(I,1)=-COS(Y(I/2,3)+U(1)-ALPHA(J))*COS(DELTA(J))
      A(I,3)=-COS(U(3)+Y(I/2,1))*COS(Y(I/2,2))
      J=J+1
   20 CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE BMAT(B,Y,U,KK,K,ALPHA,DELTA)
C     BMAT constructs the matrix B
      DIMENSION B(2*KK,3*KK),Y(KK,3),U(3),ALPHA(KK),DELTA(KK)
      I=0
      DO 10 J=1,2*K-1,2
      B(J,I*3+1)=0.
      B(J,I*3+2)=-COS(Y(I+1,2))
      B(J,I*3+3)=-COS(U(2))*SIN(Y(I+1,3)+U(1)-ALPHA(I+1))
     + *COS(DELTA(I+1))
      I=I+1
   10 CONTINUE
      I=0
      DO 20 J=2,2*K,2
      B(J,I*3+1)=-COS(U(3)+Y(I+1,1))*COS(Y(I+1,2))
      B(J,I*3+2)=SIN(U(3)+Y(I+1,1))*SIN(Y(I+1,2))
      B(J,I*3+3)=-COS(Y(I+1,3)+U(1)-ALPHA(I+1))*COS(DELTA(I+1))
      I=I+1
   20 CONTINUE
      RETURN
      END
```

```
C
C
C
      SUBROUTINE MATMUL(RM1,NDZ1,NZ1,NDS1,NS1,RM2,NDZ2,NZ2,
     + NDS2,NS2,RM3,NDZ3,NZ3,NDS3,NS3)
C    MATMUL executes the product of two matrices
      DIMENSION RM1(NDZ1,NDS1),RM2(NDZ2,NDS2),RM3(NDZ3,NDS3)
      DO 10 I=1,NZ3
      DO 10 J=1,NS3
      RM3(I,J)=0.
      DO 10 II=1,NS1
      RM3(I,J)=RM3(I,J)+RM1(I,II)*RM2(II,J)
   10 CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE INVER2 (B,N,NUM,LOG,D,NA)
C    INVER2 executes the matrix inversion
      DIMENSION B(NA,NA),NUM(NA),D(NA)
C
      CALL NORMEN (B,N,N,ZB,SB,QB,AB,NA)
C
      DO 1 I=1,N
    1 NUM(I)=0
      DET=1.
      DO 2 L=1,N
      DO 3 M=1,N
      IF (NUM(M).NE.0) GOTO 5
      D(M) = ABS (B(M,M))
      GOTO 3
    5  D(M) = 0.
    3 CONTINUE
      I1 =.1
      DO 6 I=2,N
      IF (D(I1).LT.D(I)) I1=I
    6 CONTINUE
      IF (D(I1).GE.1.E-30) GOTO 31
      NUM (I1) = L
      GOTO 2
   31 NUM (I1) = L
      DO 10 J=1,N
      IF (J.EQ.I1) GOTO 10
      DO 13 I=1,N
      IF (I.NE.I1) GOTO 20
      D(I1) = B(I1,J)/B(I1,I1)
      GOTO 13
   20   D(I) = B(I,J)-B(I,I1)*B(I1,J)/B(I1,I1)
   13   CONTINUE
      DO 14 I=1,N
   14 B(I,J) = D(I)
```

```
   10 CONTINUE
      H = B(I1,I1)
      DET = DET*H
      DO 15 I=1,N
      IF (I.NE.I1) GOTO 17
      B(I,I1) = 1/H
      GOTO 15
   17 B(I,I1) = -B(I,I1)/H
   15 CONTINUE
    2 CONTINUE
      K=0
      DO 18 I=1,N
   18 K=K+NUM(I)
      KV = (N*(N+1))/2
      LOG = .FALSE.
      IF (K.EQ.KV) LOG=.TRUE.
C
      CALL NORMEN (B,N,N,ZB1,SB1,QB1,AB1,NA)
      IF (N.GE.6) D(6)=AB*AB1
      IF (N.GE.5) D(5)=QB*QB1
      IF (N.GE.4) D(4)=SB*SB1
      IF (N.GE.3) D(3)=ZB*ZB1
      IF (N.GE.2) D(2)=1/DET
      D(1)=DET
      RETURN
      END
C
C
C
      SUBROUTINE NORMEN (A,N,M,Z,S,Q,AG,NA)
      DIMENSION A(NA,NA)
C
      Z=0
      DO 1 J=1,N
      ZS=0
      DO 2 K=1,M
    2 ZS=ZS+ABS (A(J,K))
      IF (ZS.GT.Z) Z=ZS
    1 CONTINUE
C
      S=0
      DO 3 K=1,M
      SS=0
      DO 4 J=1,N
    4 SS=SS+ABS(A(J,K))
      IF (SS.GT.S) S=SS
    3 CONTINUE
C
      Q=0
      DO 5 J=1,N
      DO 5 K=1,M
    5 Q=Q+A(J,K)**2
```

```
      Q=SQRT (Q)
C
      AG=0.
      DO 6 J=1,N
      DO 6 K=1,M
      AB = ABS (A(J,K))
      IF (AG.LT.AB) AG=AB
    6 CONTINUE
      RETURN
      END
```